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Well-Posedness of Multipoint Elliptic-Parabolic Differential Problems

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ABSTRACT

In the present work, we study the multipoint nonlocal boundary value problem for the elliptic-parabolic equation. The well-posedness of this problem in H¨older spaces with a weight is established. In applications, the coercivity inequalities for the solutions of the multipoint mixed nonlocal boundary value problems for elliptic-parabolic equations are obtained.

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1. INTRODUCTION

Methods of solutions of nonlocal boundary value problems for elliptic-parabolic differential equations have been studied extensively by many researchers (see Bazarov and Soltanov (1995), Ashyralyev and Soltanov (1995), Ashyralyev (2006) and Ashyralyev and Gercek (2008)).

In the present paper, the multipoint nonlocal boundary value problem

$$\begin{cases} -\dfrac{d^2u(t)}{dt^2} + Au(t) = g(t) \quad (0 \le t \le 1), \\ \dfrac{du(t)}{dt} - Au(t) = f(t) \quad (-1 \le t \le 0), \\ u(1) = \sum_{i=1}^{J} \alpha_i u(\lambda_i) + \varphi, \ -1 \le \lambda_i < \cdots < \lambda_J \le 0 \end{cases} \tag{1}$$

in a Hilbert space H with self-adjoint positive definite operator A is considered under assumption

$$\sum_{i=1}^{J} |\alpha_i| \le 1. \tag{2}$$

We establish the well-posedness of multipoint nonlocal boundary value problem in Hölder spaces with a weight and coercivity inequalities in Hölder norms for solutions of nonlocal boundary value problems for eliptic-parabolic equations are obtained.

2. WELL-POSEDNESS

Throughout the paper, H is a Hilbert space and A is a positive definite self-adjoint operator with $A \ge \delta I$ for some $\delta > \delta_0 > 0$, where I is the identity operator. We denote $B = A^{\frac{1}{2}}$. First of all, let us give some lemmas that will be needed below.

Lemma 1. (Sobolevskii (1977)). The following estimates hold

$$\begin{cases} \left\| B^{\alpha} e^{-tB} \right\|_{H \to H} \le t^{-\alpha} \left(\dfrac{\alpha}{e} \right)^{\alpha}, \quad 0 \le \alpha \le e, \ t > 0, \\ \left\| A^{\alpha} e^{-tA} \right\|_{H \to H} \le t^{-\alpha} \left(\dfrac{\alpha}{e} \right)^{\alpha}, \quad 0 \le \alpha \le e, \ t > 0, \\ \left\| \left(I - e^{-2B} \right)^{-1} \right\|_{H \to H} \le M(\delta) \end{cases} \tag{3}$$

for some $M(\delta) \ge 0$.

Lemma 2. Assume that (2) holds. Then, the operator

$$B\left(I-e^{-2B}\right)+I+e^{-2B}-2\sum_{i=1}^{J}\alpha_i e^{-(B-\lambda_i A)}$$

has an inverse

$$T=\left(B\left(I-e^{-2B}\right)+I+e^{-2B}-2\sum_{i=1}^{J}\alpha_i e^{-(B-\lambda_i A)}\right)^{-1}$$

and the following estimates are satisfied

$$\|T\|_{H\to H}\le M(\delta),\ \|BT\|_{H\to H}\le M(\delta). \tag{4}$$

A function $u(t)$ is called a *solution* of problem (1) if the following conditions are satisfied:

(i) $u(t)$ is twice continuously differentiable on the segment $(0,1]$ and continuously differentiable on the segment $[-1,1]$,
(ii) The element $u(t)$ belongs to domain $D(A)$ of A for all $t\in[-1,1]$ and the function $Au(t)$ is continuous on the segment $[-1,1]$,
(iii) $u(t)$ satisfies the equations and the nonlocal boundary condition (1).

A solution of problem (1) defined in this manner will henceforth be referred as a solution of problem (1) in the space $C(H)=C([-1,1],H)$ of all continuous functions $\varphi(t)$ defined on $[-1,1]$ with values in H equipped with the norm

$$\|\varphi\|_{C([-1,1],H)}=\max_{-1\le t\le 1}\|\varphi(t)\|_H .$$

Now, we will obtain the formula for solution of problem (1). It is known that (see Krein (1966)) for smooth data of the problems

$$\begin{cases}-u''(t)+Au(t)=g(t), & (0\le t\le 1),\\ u(0)=u_0,\ u(1)=u_1,\end{cases} \tag{5}$$

$$\begin{cases} u'(t) - Au(t) = f(t), \quad (-1 \le t \le 0), \\ u(0) = u_0, \end{cases} \tag{6}$$

there are unique solutions of problems (5), (6) and the following formulas hold

$$\begin{aligned} u(t) = \left(I - e^{-2B}\right)^{-1} \Bigg[\left[\left(e^{-tB} - e^{-(-t+2)B}\right) u_0 + \left(e^{-(1-t)B} - e^{-(t+1)B}\right) u_1 \right] \\ + \left(e^{-(1-t)B} - e^{-(t+1)B}\right) (2B)^{-1} \int_0^1 \left(e^{-(1-s)B} - e^{-(s+1)B}\right) g(s) ds \Bigg] \\ -(2B)^{-1} \int_0^1 \left(e^{-(t+s)B} - e^{-|t-s|B}\right) g(s) ds, \quad 0 \le t \le 1, \end{aligned} \tag{7}$$

$$u(t) = e^{tA_{u_0}} + \int_0^t e^{(t-s)A} f(s) ds, \quad -1 \le t \le 0. \tag{8}$$

Using the condition $u(1) = \sum_{i=1}^{J} \alpha_1 u(\lambda_i) + \varphi$ and formulas (7), (8), we can write

$$\begin{aligned} u(t) = \left(I - e^{-2B}\right)^{-1} \left[\left(e^{-tB} - e^{-(-t+2)B}\right) u_0 + \left(e^{-(1-t)B} - e^{-(t+1)B}\right) \right] \\ \times \Bigg[\left(\sum_{i=1}^{J} \left[\alpha_i e^{\lambda_k A} u_0 + \int_0^{\lambda_i} e^{(\lambda_i - s)A} f(s) ds \right] + \varphi \right) \\ +(2B)^{-1} \int_0^1 \left(e^{-(1-s)B} - e^{-(s+1)B}\right) g(s) ds \Bigg] \\ -(2B)^{-1} \int_0^1 \left(e^{-(t+s)B} - e^{-|t-s|B}\right) g(s) ds, \quad 0 \le t \le 1. \end{aligned} \tag{9}$$

For u_0, using the condition $u'(0+) = Au(0) + f(0)$ and formula (9), we obtain the operator equation

$$Au(0) + f(0) = \left(I - e^{-2B}\right)^{-1} \left[-B\left(I + e^{-2B}\right) u_0 \right. \tag{10}$$

$$+2Be^{-B}\left(\sum_{k=1}^{n}\alpha_k e^{\lambda_i A}u_0+\sum_{i=1}^{J}\alpha_i\int_0^{\lambda_i}e^{(\lambda_i-s)A}f(s)ds+\varphi\right)\Bigg]\Bigg]$$

$$+\left(I-e^{-2B}\right)^{-1}e^{-B}\int_0^1\left(e^{-(1-s)B}-e^{-(s+1)B}\right)g(s)ds+\int_0^1 e^{-sB}g(s)ds.$$

Since the operator

$$B\left(I-e^{-2B}\right)+I+e^{-2B}-2\sum_{i=1}^{J}\alpha_i e^{-(B-\lambda_i A)}$$

has an inverse

$$T=\left(B\left(I-e^{-2B}\right)+I+e^{-2B}-2\sum_{i=1}^{J}\alpha_i e^{-(B-\lambda_i A)}\right)^{-1},$$

for the solution of operator equation (10) we have the formula

$$u_0=T\left[e^{-B}\left[2\sum_{i=1}^{J}\alpha_i\int_0^{\lambda_i}e^{(\lambda_i-s)A}f(s)ds\right.\right. \tag{11}$$
$$\left.+\int_0^1 B^{-1}\left(e^{-(1-s)A^{\frac{1}{2}}}-e^{-(s+1)A^{\frac{1}{2}}}\right)g(s)ds\right]+2e^{-B}\varphi\Bigg]$$
$$+\left(I-e^{-2B}\right)TB^{-1}\left[-f(0)+\int_0^1 e^{-sB}g(s)ds\right].$$

Hence, for the solution of the nonlocal boundary value problem (1), we have formulas (7), (8) and (11).

For $\alpha\in(0,1)$, let $C_{0,1}^{\alpha}([-1,1],H)$, $C_{0,1}^{\alpha}([0,1],H)$, and $C_{0,1}^{\alpha}([-1,0],H)$ denote the Banach spaces obtained by the completion of the set of all smooth $H-$valued function $\varphi(t)$ defined respectively on $[-1,1]$, $[0,1]$ and $[-1,0]$ with the norms

$$\|\varphi\|_{C_{0,1}^{\alpha}([-1,1],H)}=\|\varphi\|_{C([-1,1],H)}+\sup_{-1<t<t+\tau<0}\frac{(-t)^{\alpha}\|\varphi(t+\tau)-\varphi(t)\|_H}{\tau^{\alpha}}$$

$$+\sup_{0<t<t+\tau<1}\frac{(1-t)^{\alpha}(t+\tau)^{\alpha}\left\|\varphi(t+\tau)-\varphi(t)\right\|_{H}}{\tau^{\alpha}},$$

$$\left\|\varphi\right\|_{C_{0,1}^{\alpha}([0,1],H)}=\left\|\varphi\right\|_{C([0,1],H)}+\sup_{0<t<t+\tau<1}\frac{(1-t)^{\alpha}(t+\tau)^{\alpha}\left\|\varphi(t+\tau)-\varphi(t)\right\|_{H}}{\tau^{\alpha}},$$

$$\left\|\varphi\right\|_{C_{0}^{\alpha}([-1,0],H)}=\left\|\varphi\right\|_{C([-1,0],H)}+\sup_{-1<t<t+\tau<0}\frac{(-t)^{\alpha}\left\|\varphi(t+\tau)-\varphi(t)\right\|_{H}}{\tau^{\alpha}}.$$

Here, $C([a,b],H)$ stands for the Banach space of all continuous functions $\varphi(t)$ defined on $[a,b]$ with values in H equipped with the norm

$$\left\|\varphi\right\|_{C([a,b],H)}=\max_{a\le t\le b}\left\|\varphi(t)\right\|_{H}.$$

We say that problem (1) is *well-posed* in $C(H)$, if there exists a unique solution $u(t)$ in $C(H)$ of problem (1) for any $g(t)\in C([0,1],H)$, $f(t)\in C([-1,0],H)$ and $\varphi\in D(A)$ and also the following coercivity inequality is satisfied

$$\left\|u''\right\|_{C([0,1],H)}+\left\|u'\right\|_{C([-1,0],H)}+\left\|Au\right\|_{C(H)} \tag{12}$$

$$\le M\left[\left\|g\right\|_{C([0,1],H)}+\left\|f\right\|_{C([-1,0],H)}+\left\|A\varphi\right\|_{H}\right],$$

where M is independent of $\varphi, f(t)$ and $g(t)$.

Problem (1) is not well-posed in $C(H)$ (Ashyralyev and Soltanov (1995)). The well-posedness of boundary value problem (1) can be established if one considers this problem in certain spaces $F(H)$ of smooth H-valued functions on $[-1,1]$.

A function $u(t)$ is said to be a *solution* of problem (1) in $F(H)$, if it is a solution of this problem in $C(H)$ and the functions $u''(t)(t\in[0,1])$, $u'(t)(t\in[-1,1])$ and $Au(t)(t\in[-1,1])$ belong to $F(H)$.

As in the case of the space $C(H)$, we say that problem (1) is well-posed in $F(H)$, if the following coercivity inequality is satisfied

$$\|u''\|_{F([0,1],H)} + \|u'\|_{F([-1,0],H)} + \|Au\|_{F(H)} \tag{13}$$
$$\leq M\left[\|g\|_{F([0,1],H)} + \|f\|_{F([-1,0],H)} + \|A\varphi\|_H\right],$$

where M does not depend on $\varphi, f(t)$ and $g(t)$.

If we set $F(H)$, equal to $C_{0,1}^{\alpha}(H) = C_{0,1}^{\alpha}([-1,1],H)$ $(0<\alpha<1)$ then we can establish our main theorem.

Theorem 1. Suppose $\varphi \in D(A)$. Then, boundary value problem (1) is well-posed in the Hölder space $C_{0,1}^{\alpha}(H)$ and the following coercivity inequality holds

$$\|u''\|_{C_{0,1}^{\alpha}([0,1],H)} + \|u'\|_{C_0^{\alpha}([-1,0],H)} + \|Au\|_{C_{0,1}^{\alpha}(H)} \tag{14}$$
$$\leq M(\delta)\left[\frac{1}{\alpha(1-\alpha)}\left[\|f\|_{C_0^{\alpha}([-1,0],H)} + \|g\|_{C_{0,1}^{\alpha}([0,1],H)}\right] + \|A\varphi\|_H\right].$$

Here, $M(\delta)$ is independent of $f(t)$, $g(t)$ and φ.

Proof. Coercivity inequality (14) is based on the estimate

$$\|u'\|_{C_0^{\alpha}([-1,0],H)} + \|Au\|_{C_0^{\alpha}([-1,0],H)} \leq \frac{M(\delta)}{\alpha(1-\alpha)}\|f\|_{C_0^{\alpha}([-1,0],H)} + M + \|Au_0\|_H \tag{15}$$
$$\leq \frac{M(\delta)}{\alpha(1-\alpha)}\|f\|_{C_0^{\alpha}([-1,0],H)} M\,\|Au_0\|_H$$

for the solution of inverse Cauchy problem (6) and on the estimate

$$\|u''\|_{C_{0,1}^{\alpha}([0,1],H)} + \|Au\|_{C_{0,1}^{\alpha}([0,1],H)} \leq \frac{M(\delta)}{\alpha(1-\alpha)}\|g\|_{C_{0,1}^{\alpha}([0,1],H)} \tag{16}$$
$$+M(\delta)\left[\|Au_0\|_H + \|Au_1\|_H\right]$$

for the solution of boundary value problem (5), and on the estimates

$$\|Au_0\|_H \leq M(\delta)\left[\|f\|_{C_0^{\alpha}([-1,0],H)} + \|g\|_{C_{0,1}^{\alpha}([0,1],H)} + \|A\varphi\|_H\right], \tag{17}$$

$$\|Au_1\|_H \le \frac{M(\delta)}{\alpha(1-\alpha)}\left[\|f\|_{C_0^{\alpha}([-1,0],H)} + \|g\|_{C_{0,1}^{\alpha}([0,1],H)}\right] + M(\delta)\|A\varphi\|_H \quad (18)$$

for the solution of boundary value problem (1). Estimates (15), (16) were established in Sobolevskii (1977). Let us obtain estimates (17), (18). Applying formulas (7), (8), and (11), we get

$$\begin{aligned} Au_0 &= 2Te^{-B}\sum_{i=1}^{J}\alpha_i\int_{\lambda_i}^{0} Ae^{(\lambda_i-s)A}\left(f(s)-f(\lambda_i)\right)ds \qquad (19)\\ &+Te^{-B}\left[\int_0^1 Be^{-(1-s)B}\left(g(s)-g(1)\right)ds + \int_0^1 Be^{-(s+1)B}\left(g(s)-g(0)\right)ds\right]\\ &+2Te^{-B}A\varphi + \left(I-e^{-2B}\right)T\int_0^1 Be^{-sA^{\frac{1}{2}}}\left(g(s)-g(0)\right)ds\\ &+2Te^{-B}\sum_{k=1}^{n}\alpha_k\left(e^{\lambda_i A}-I\right)f(\lambda_i)\\ &+T\left(e^{-B}-e^{-2B}\right)g(1)+T\left(I+2e^{-3B}-2e^{-2B}-e^{-B}\right)g(0)\\ &+TB\left(e^{-2B}-I\right)f(0) = J_1+J_2+J_3+J_4+J_5+J_6, \end{aligned}$$

where

$$J_1 = 2Te^{-B}\sum_{i=1}^{J}\alpha_i\int_0^{\lambda_i} Ae^{(\lambda_i-s)A}\left(f(s)-f(\lambda_i)\right)ds + 2Te^{-B}A\varphi,$$

$$J_2 = Te^{-B}\left[\int_0^1 Be^{-(1-s)B}\left(g(s)-g(1)\right)ds + \int_0^1 Be^{-(s+1)B}\left(g(s)-g(0)\right)ds\right],$$

$$J_3 = \left(I-e^{-2B}\right)T\int_0^1 Be^{-sB}\left(g(s)-g(0)\right)ds,$$

$$J_4 = 2Te^{-B}\sum_{i=1}^{J}\alpha_i\left(e^{\lambda_i A}-I\right)f(\lambda_i),$$

$$J_5 = T\left(e^{-B}-e^{-2B}\right)g(1)+T\left(I+2e^{-3B}-2e^{-2B}-e^{-B}\right)g(0),$$

$$J_6 = TB\left(e^{-2B}-I\right)f(0),$$

$$Au_1 = \sum_{i=1}^{J} \alpha_i e^{\lambda_i A} Au_0 + \sum_{i=1}^{J} \alpha_i \int_{\lambda_i}^{0} Ae^{(\lambda_i - s)A} \left(f(s) - f(\lambda_i) \right) ds \tag{20}$$

$$+\sum_{i=1}^{J} \alpha_i \left(I - e^{\lambda_i A} \right) f(\lambda_i) + A\varphi = K_1 + K_2 + K_3,$$

where

$$K_1 = \sum_{i=1}^{J} \alpha_i e^{\lambda_i A} Au_0, \;\; K_2 = \sum_{i=1}^{J} \alpha_i \int_{\lambda_k}^{0} Ae^{(\lambda_i - s)A} \left(f(s) - f(\lambda_i) \right) ds,$$

$$K_3 = \sum_{i=1}^{J} \alpha_i \left(I - e^{\lambda_i A} \right) f(\lambda_i) + A\varphi.$$

First, we obtain (17). Let us estimate norm of J_k for $k = 1, \cdots, 6$ separately to establish the estimate for norm of (19). Using the triangle inequality, assumption (2), estimates (3), (4), and the definition of the norm of the space $C_0^{\alpha}([-1,0], H)$, we obtain

$$\|J_1\|_H \leq \|T\|_{H \to H} \left\| B^2 e^{-B} \right\|_{H \to H} 2 \sum_{i=1}^{J} |\alpha_i| \int_{\lambda_k}^{0} \left\| e^{-(s-\lambda_k)A} \right\|_{H \to H} \|f(s) - f(\lambda_k)\|_H \; ds$$

$$+2\|T\|_{H \to H} \left\| e^{-B} \right\|_{H \to H} \|A\varphi\|_H \leq M_1(\delta) \Big[\|f\|_{C_0^{\alpha}([-1,0],H)} + \|A\varphi\|_H \Big].$$

Let us estimate the norm of J_2. Using the triangle inequality, estimates (3), (4), and the definition of the norm of the space $C_{0,1}^{\alpha}([0,1], H)$, we get

$$\|J_2\|_H \leq \|BT\|_{H \to H} \int_0^1 \left\| e^{-(2-s)B} \right\|_{H \to H} \|g(s) - g(1)\|_H \; ds$$

$$+\|BT\|_{H \to H} \int_0^1 \left\| e^{-(s+2)B} \right\|_{H \to H} \|g(s) - g(0)\|_H \, ds$$

$$\leq M_2(\delta) \int_0^1 [(1-s)^{\alpha} + s^{\alpha}] \; ds \|g\|_{C_{0,1}^{\alpha}([0,1],H)} \leq M_2(\delta) \|g\|_{C_{0,1}^{\alpha}([0,1],H)}.$$

We shall now estimate the norm of J_3. From the triangle inequality, estimates (3), (4) and the definition of the norm of the space $C_{0,1}^{\alpha}([0,1], H)$

it follows that

$$\|J_3\|_H \le \left\{1 + \left\|e^{-2B}\right\|_{H\to H}\right\}\|BT\|_{H\to H}\int_0^1 \left\|e^{-sB}\right\|_{H\to H}\|g(s)-g(0)\|_H \; ds$$
$$\le M_3(\delta)\int_0^1 s^{\alpha} ds \|g\|_{C_{0,1}^{\alpha}([0,1],H)} \le M_3(\delta)\|g\|_{C_{0,1}^{\alpha}([0,1],H)}.$$

Now, we will estimate the norm of J_4. Using the triangle inequality, assumption (2), the definition of the norm of the space $C_0^{\alpha}([-1,0],H)$ and estimates (3), (4), we obtain

$$\|J_4\|_H \le \|T\|_{H\to H}\left\|e^{-B}\right\|_{H\to H} 2\sum_{i=1}^{J}|\alpha_i|\left(1+\left\|e^{\lambda_i A}\right\|_{H\to H}\right)\|f(\lambda_i)\|_H$$
$$\le M_4(\delta)\max_{-1\le t\le 0}\|f(t)\|_H \le M_4(\delta)\|f\|_{C_0^{\alpha}([-1,0],H)}.$$

It follows from the triangle inequality, the definition of the norm of $C_{0,1}^{\alpha}([0,1],H)$ and estimates (3), (4) that

$$\|J_5\|_H \le \|T\|_{H\to H}\Big[\left(\left\|e^{-2B}\right\|_{H\to H} + \left\|e^{-B}\right\|_{H\to H}\right)\|g(1)\|_H$$
$$+\left(1+2\left\|e^{-3B}\right\|_{H\to H} + 2\left\|e^{-2B}\right\|_{H\to H} + \left\|e^{-B}\right\|_{H\to H}\right)\|g(0)\|_H\Big]$$
$$\le M_5(\delta)\max_{0\le t\le 1}\|g(t)\|_H \le M_5(\delta)\|g\|_{C_{0,1}^{\alpha}([0,1],H)}.$$

Finally, using estimates (3), (4), the definition of the norm of the space $C_0^{\alpha}([-1,0],H)$, we get

$$\|J_6\|_H \le \|BT\|_{H\to H}\left(1+\left\|e^{-2B}\right\|_{H\to H}\right)\|f(0)\|_H$$
$$\le M_6(\delta)\max_{-1\le t\le 0}\|f(t)\|_H \le M_6(\delta)\|f\|_{C_0^{\alpha}([-1,0],H)}.$$

Hence, combining the estimates for the norm of J_k, $k=1,\cdots,6$, we obtain (17).

Second, let us obtain (18). Now, let us estimate norm of K_1, K_2 and K_3 separately to establish the estimate for the norm of (20).

From the triangle inequality, assumption (2) and estimates (3), (17) it follows that

$$\|K_1\|_H \le \sum_{i=1}^{J} |\alpha_i| \left\| e^{\lambda_i A} \right\|_{H \to H} \|Au_0\|_H$$
$$\le \|Au_0\|_H \le M_1(\delta)\left[\|g\|_{C_{0,1}^{\alpha}([0,1],H)} + \|f\|_{C_0^{\alpha}([-1,0],H)} + \|A\varphi\|_H \right].$$

Now, we will estimate for the norm of K_2. Using the triangle inequality, assumption (2), and estimates (3), (4), we get

$$\|K_2\|_H \le \sum_{i=1}^{J} |\alpha_i| \int_{\lambda_i}^{0} \left\| Ae^{\lambda_i A} \right\|_{H \to H} \|f(s) - f(\lambda_i)\|_H \; ds$$
$$\le M_2(\delta) \int_{\lambda_k}^{0} \frac{(s-\lambda_i)^{\alpha} ds}{(s-\lambda_i)(-s)^{\alpha}} \|f\|_{C_0^{\alpha}([-1,0],H)} \le \frac{M_2(\delta)}{\alpha(1-\alpha)} \|f\|_{C_0^{\alpha}([-1,0],H)}.$$

Finally, using the triangle inequality, assumption (2), estimate (3), and the definition of the norm of the space $C_0^{\alpha}([-1,0],H)$, we obtain

$$\|K_3\|_H \le \sum_{i=1}^{J} |\alpha_i| \left\| I - e^{\lambda_i A} \right\|_{H \to H} \|f(\lambda_i)\|_H + \|A\varphi\|_H$$
$$\le M_3(\delta) \max_{-1 \le t \le 0} \|f(\lambda_i)\|_H + \|A\varphi\|_H \le M_3(\delta) \|f\|_{C_0^{\alpha}([-1,0],H)} + \|A\varphi\|_H .$$

Thus, combining these three estimates for the norms of K_1, K_2 and K_3, we obtain (18). This concludes the proof of the Theorem 2.

3. APPLICATIONS

Now, let us consider the applications of Theorem 1. First, the multipoint nonlocal boundary value problem for elliptic-parabolic equation

$$\begin{cases} -u_{tt}-(a(x)u_x)_x+\delta u=g(t,x), \quad 0<t<1, \quad 0<x<1, \\ u_t+(a(x)u_x)_x-\delta u=f(t,x), \quad -1<t<1, \quad 0<x<1, \\ u(t,0)=u(t,1), \ u_x(t,0)=u_x(t,1), \quad -1\le t\le 1, \\ u(1,x)=\sum_{i=1}^{J}\alpha_i u(\lambda_i,x)+\varphi(x), \ \sum_{i=1}^{J}|\alpha_i|\le 1, \\ -1\le \lambda_1<\lambda_2<\cdots<\lambda_i<\cdots<\lambda_J\le 0, \ 0\le x\le 1, \\ u(0+,x)=u(0-,x), \ u_t(0+,x)=u_t(0-,x), \ 0\le x\le 1 \end{cases} \tag{21}$$

is considered. Problem (21) has a unique smooth solution $u(t,x)$ for the smooth $g(t,x)(t\in[0,1],x\in[0,1])$, $f(t,x)(t\in[-1,0],x\in[0,1])$, $a(x)\ge a>0(x\in(0,1))$ functions and $\delta=\text{const}>0$.

We introduce the Hilbert space $L_2[0,1]$ of all the square integrable functions defined on $[0,1]$, $W_2^1[0,1]$ and $W_2^2[0,1]$ equipped with the norms

$$\|\varphi\|_{W_2^1[0,1]}=\|\varphi\|_{L_2[0,1]}+\left(\int_0^1|\varphi_x|^2\,dx\right)^{1/2},$$

$$\left\|\varphi^h\right\|_{W_2^2[0,1]}=\|\varphi\|_{L_2[0,1]}+\left(\int_0^1|\varphi_x|^2\,dx\right)^{1/2}+\left(\int_0^1|\varphi_{xx}|^2\,dx\right)^{1/2}.$$

This allows us to reduce mixed prob;lem (21) to the nonlocal boundary value problem (1) Hilbert space $H=L_2[0,1]$ with a self-adjoint positive definite operator A defined by (21).

Theorem 2. The solutions of nonlocal boundary value problem (21) satisfy the coercivity inequality

$$\begin{aligned} &\|u_{tt}\|_{C_{0,1}^{\alpha}([0,1],L_2[0,1])}+\|u_t\|_{C_0^{\alpha}([-1,0],L_2[0,1])}+\|u\|_{C_{0,1}^{\alpha}([-1,1],W_2^2[0,1])} \\ &\le \frac{M(\delta)}{\alpha(1-\alpha)}\left[\|g\|_{C_{0,1}^{\alpha}([0,1],L_2[0,1])}+\|f\|_{C_0^{\alpha}([-1,0],L_2[0,1])}\right]+M(\delta)\|\varphi\|_{W_2^2[0,1]}. \end{aligned}$$

Here, $M(\delta)$ does not depend on $f(t,x), g(t,x)$ and $\varphi(x)$.

The proof of Theorem 2 is based on the abstract Theorem 1 and the symmetry properties of the space operator generated by problem (21).

Second, let Ω be the unit open cube in the n-dimensional Eucledean space $\mathbb{R}^n$ $(0 < x_k < 1,\ 1 \le k \le n)$ with boundary S, $\overline{\Omega} = \Omega \cup S$.

In $[-1,1]\times\Omega$, the multipoint mixed boundary value problem for multidimensional mixed equation

$$\begin{cases} -u_{tt} - \sum_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} = g(t,x), \quad 0 < t < 1, \quad x \in \Omega, \\ u_t + \sum_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} = f(t,x), \quad -1 < t < 0, \quad x \in \Omega, \\ u(t,x) = 0, \ x \in S, \quad -1 \le t \le 1, \\ u(1,x) = \sum_{i=1}^{J} \alpha_i u(\lambda_i, x) + \varphi(x), \ \sum_{i=1}^{J} |\alpha_i| \le 1, \\ -1 \le \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots < \lambda_J \le 0, \\ u(0+,x) = u(0-,x), \ u_t(0+,x) = u_t(0-,x), \ x \in \overline{\Omega} \end{cases} \tag{22}$$

is considered.

Here, $a_r(x)$ $(x \in \Omega)$, $g(t,x)$ $(t \in (0,1), x \in \overline{\Omega})$ and $f(t,x)$ $(t \in (-1,0), x \in \overline{\Omega})$ are given smooth functions and $a_r(x) \ge a > 0$.

We introduce the Hilbert space $L_2(\overline{\Omega})$ of all the square integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\|\varphi\|_{L_2(\overline{\Omega})} = \sqrt{\int \cdots \int_{x \in \overline{\Omega}} |\varphi(x)|^2 \, dx_1 \cdots dx_n}$$

And the Hilbert spaces $W_2^1(\overline{\Omega})$, $W_2^2(\overline{\Omega})$ defined on $\overline{\Omega}$, equipped with the norms

$$\|\varphi\|_{W_2^1(\overline{\Omega})} = \|\varphi\|_{L_2(\overline{\Omega})} + \sqrt{\int \cdots \int_{x \in \overline{\Omega}} \sum_{r=1}^{n} |\varphi_{x_r}|^2 \, dx_1 \cdots dx_n},$$

$$\left\|\varphi^h\right\|_{W_2^2(\overline{\Omega})}=\left\|\varphi^h\right\|_{L_{2h}}+\sqrt{\int\cdots\int_{x\in\overline{\Omega}}\sum_{r=1}^{n}\left|\varphi_{x_r}\right|^2dx_1\cdots dx_n}$$
$$+\sqrt{\int\cdots\int_{x\in\overline{\Omega}}\sum_{r=1}^{n}\left|\varphi_{x_r\overline{x_r}}\right|^2dx_1\cdots dx_n}.$$

Problem (22) has a unique smooth solution $u(t,x)$ for the smooth functions $a_r(x)$, $g(t,x)$ and $f(t,x)$. This allows us to reduce mixed problem (22) to nonlocal boundary value problem (1) in Hilbert space $H=L_2(\overline{\Omega})$ with a self-adjoint positive definite operator A defined by (22).

Theorem 3. The solutions of nonlocal boundary value problem (22) satisfy the coercivity inequality

$$\left\|u_{tt}\right\|_{C_{0,1}^{\alpha}([0,1],L_2(\overline{\Omega}))}+\left\|u_t\right\|_{C_0^{\alpha}([-1,0],L_2(\overline{\Omega}))}+\left\|u\right\|_{C_{0,1}^{\alpha}([-1,1],W_2^2(\overline{\Omega}))}$$
$$\leq\frac{M(\delta)}{\alpha(1-\alpha)}\left[\left\|g\right\|_{C_{0,1}^{\alpha}([0,1],L_2(\overline{\Omega}))}+\left\|f\right\|_{C_0^{\alpha}([-1,0],L_2(\overline{\Omega}))}\right]+M(\delta)\left\|\varphi\right\|_{W_2^2(\overline{\Omega})}.$$

The proof of Theorem 3 is based on the abstract Theorem 1 and the symmetry properties of the space operator generated by problem (22) and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 4. (Sobolevskii (1975)). For the solution of the elliptic differential problem

$$\sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r}=\omega(x),\quad x\in\Omega,\ u(x)=0,\ x\in S$$

the following coercivity inequality holds

$$\sum_{r=1}^{n}\left\|u_{x_rx_r}\right\|_{L_2(\Omega))}\leq M\left\|\omega\right\|_{L_2(\Omega))}.$$

Here M is independent of $w(x)$.

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